

Nonlinear Hydrodynamics from Flow of Retarded Green's Function

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We study the radial flow of retarded Green's function of energy-momentum tensor and R -current of dual gauge theory in presence of generic higher derivative terms in bulk Lagrangian. These are first order non-linear *Riccati* equations. We solve these flow equations analytically and obtain second order transport coefficients of boundary plasma. This way of computing transport coefficients has an advantage over usual Kubo approach. The non-linear equation turns out to be a linear first order equation when we study the Green's function perturbatively in momentum. We consider several examples including $Weyl^4$ term and generic four derivative terms in bulk. We also study the flow equations for R -charged black holes and obtain exact expressions for second order transport coefficients for dual plasma in presence of arbitrary chemical potentials. Finally we obtain higher derivative corrections to second order transport coefficients of boundary theory dual to five dimensional gauge supergravity.

CONTENTS

I. Introduction and Discussion	1	D. Functions appeared in four derivative action	18
II. Flow of retarded Green's function of energy-momentum tensor	3	E. Functions appeared in Higher-derivative Charged black-hole action	18
A. A renormalized response function	5	References	18
B. Second order transport coefficients from flow equation	5		
III. Higher derivative correction to flow equation	6		
IV. Examples: String theory corrections to flow equation	7		
A. $Weyl^4$ term	7		
B. Four derivative term	8		
1. Exact result for Gauss-Bonnet black hole	9		
V. Flow equation for charged black holes	10		
A. R -charged black holes	11		
B. Charged black holes in higher derivative gravity	12		
VI. Flow of retarded Green's function of boundary R current	13		
Acknowledgments	15		
A. Second order transport coefficients from Kubo formula	15		
B. Equivalence of Boundary Terms	16		
C. Functions appeared in String theory corrected action	18		

I. INTRODUCTION AND DISCUSSION

Fluid/gravity correspondence has become an interesting aspect of current theoretical physics research after beginning of *RHIC* program in 2000. The experimental data implies that the quark-gluon plasma (QGP) produced in *RHIC* are in a new state called thermalized matter. The evolution of QGP and hadronic matter in this state can be described by hydrodynamics. The temperature of the gas of quarks and gluons produced at RHIC is approximately 170MeV which is very close to the confinement temperature of QCD. At this high temperature they are not in the weakly coupled regime of QCD. This new phase of nuclear matter is known as the strongly coupled quark-gluon plasma (sQGP). Obviously to study their properties the usual perturbation theory does not work. One needs different approach to deal with these strongly coupled system.

The holographic hydrodynamics (or fluid/gravity correspondence) is an important tool for understanding some properties of strongly coupled CFTs in terms of the dual AdS black holes physics¹ [1]-[25].

Low frequency (long wavelength) fluctuations of any interacting quantum field theory should be described by hydrodynamics. Low energy behavior of strongly coupled field theory (with gravity dual) is governed by a weakly coupled black hole space-time in one higher dimension.

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¹ QCD is approximately conformal at sufficiently large energies.

On the other hand in classical theory of gravity a black hole can be viewed as a fictitious fluid membrane (with hydrodynamic characteristic) living on the horizon of the black hole [26]. Therefore from UV/IR point of view it would be interesting to understand the precise relation between the membrane fluid and the low frequency description of strongly coupled field theory sitting at the boundary of *AdS* black hole space-time.

One can read off the transport coefficients like shear-viscosity coefficients, of boundary plasma from its retarded Green's function of stress tensor

$$G_{xy,xy}^R(k_\mu) = -i \int dt dx e^{ik \cdot x} \langle [T_{xy}(x), T_{xy}(0)] \rangle. \quad (I.1)$$

We write the Green's function in powers of momentum and it is given by²,

$$G_{xy,xy}^R(k_\mu) = -i\eta\omega + \eta\tau_\pi\omega^2 - \frac{\kappa}{2}[(p-2)\omega^2 + q^2] + \mathcal{O}(k^3), \quad (I.2)$$

where, p is spatial dimension and $p \geq 3$ and

η : shear viscosity coefficient,

τ_π : relaxation time for shear viscous stress,

κ : a new coefficient defined in Ref[9, 10]

and ω is frequency and q is spatial momentum ($k = \{\omega, 0, 0, q\}$).

In [27] it was shown that, at low momentum ($k_\mu \rightarrow 0$), the evolution of the boundary Green's function is independent of the radial direction and hence, it can be computed either at horizon or at boundary. Computing the Green's function at black hole horizon one can show that the shear viscosity coefficient, which is a first-order transport coefficient of the boundary plasma, can be obtained from the low frequency characteristics of the membrane fluid. In [28–30], this issue has been generalized for gravity theory with generic higher-derivative interactions in arbitrary backgrounds. It has been proved that the membrane fluid does give the correct shear-viscosity coefficient of the boundary plasma in arbitrary higher derivative gravity and just the knowledge of near-horizon geometry of the dual *AdS* black hole is enough for computing the shear-viscosity coefficient [31].

To specify the boundary plasma completely one also needs to understand its higher-order transport coefficients. For this one needs to move away from the low frequency ($k_\mu \rightarrow 0$) limit. In this case, the Green's functions flow non-trivially with the radial direction and the flow depends on full black hole geometry. Although the boundary plasma and the membrane fluid have same shear-viscosity coefficients, other transport coefficients can certainly differ and it is not clear how the two are related. In Fig. 1 we plot the radial evolution of response

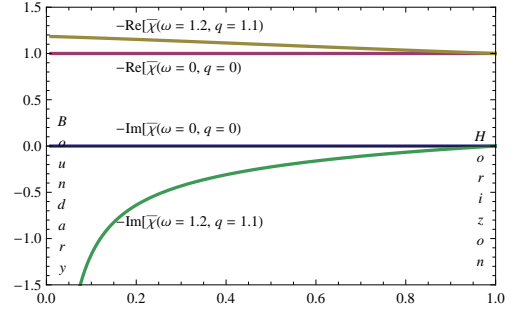


FIG. 1. Flow of Green's function from horizon to boundary for two derivative gravity.

function for two derivative gravity³. In zero frequency limit flow of the response function $\bar{\chi}$ is trivial. Its real part is constant (the constant is one in our scaling) and imaginary part is zero. For finite ω and q the response function (both real and imaginary part) has a non-trivial evolution. But the horizon value of the function for non-zero frequency is same as the horizon value of the function for zero frequency. Therefore we conclude that for two derivative gravity dual, the full momentum response at the horizon automatically corresponds to only the zero momentum limit of the boundary response.

However in presence of higher derivative terms in the action the situation is different. The full momentum response at horizon depends on spatial momentum⁴. We plot the flow of response function for *Weyl*⁴ interaction in Fig. 2.

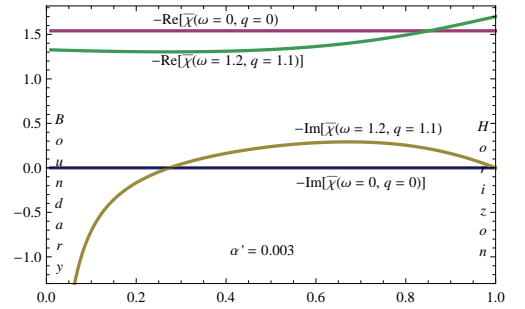


FIG. 2. Flow of Green's function from Horizon to boundary for higher derivative gravity for different values of ω and q .

From this plot we see that in presence of higher-derivative interaction the horizon value of the response function depends of k_μ unlike two derivative theory.

We also see that the imaginary part of $\bar{\chi}$ diverges at boundary. This is usual *UV* divergence. One has to add a counter term to cancel this divergence.

In this paper, we study the flow equations for retarded Green's function of boundary theory analytically and

² Here we have dropped the frequency independent part of the Green's function.

³ We define the response function $\bar{\chi} = \frac{-G_{xy,xy}^R(\omega, q)}{i\omega}$.

⁴ We will also see this analytically in sections III.

find higher order transport coefficients of the boundary plasma solving this equation. We generalize the analysis for generic higher derivative gravity theory and also for R charge black holes. The flow equation for Green's function is a first order non-linear differential equation of *Riccati* type. Because of its non-linear nature it is hard to solve this equation exactly. After a change of variable one can reduce this non-linear equation to a second order linear homogeneous differential equation. But to solve this we need to specify two boundary conditions. In this paper we deal with the non-linear equation and specify the boundary condition at the horizon. Therefore the hydrodynamic characteristic of the field theory at UV fix point is determined by IR boundary condition.

For two derivative Einstein-Hilbert action the flow equation of retarded Green's function has been derived in [27]. But it is not obvious how to generalize the flow equation for higher derivative gravity. The derivation given in [27] was based on the canonical form of graviton's action. In this paper we have considered generic higher derivatives terms in the bulk Lagrangian. Following the prescription given in [29] we construct an effective action for transverse graviton which has the canonical form in presence of any higher derivative terms in the bulk, and derive the flow equation for Green's function. Solving this flow equation perturbatively in ω and q we obtain second order transport coefficients namely τ_π (relaxation time) and κ of the dual plasma⁵. In this way of computing the transport coefficients has an advantage over usual Kubo approach. In Kubo approach, one has to first find the transverse graviton by solving a second order differential equation and then compute retarded Green's function. Instead, the flow equation is a first order differential equation (although non-linear). As we want a perturbative expansion of Green's function in powers of ω and q the equation turns out to be a linear first order differential equation. Thus, technically, it is simpler to get results for causal hydrodynamics, particularly when the dual bulk theory is complicated.

The paper is organized as follows. We have worked in five-dimensional bulk theory, the dual gauge theory is four-dimensional. In section II we review the derivation of flow equation of boundary Green's function for two derivative gravity. Then solving this flow equation we compute second order transport coefficients κ and τ_π of boundary plasma. Our results matches with previous computations of [9, 10]. In section III we present the flow equation in presence of generic higher derivative interaction. We calculate the higher derivative correction for τ_π and κ in section IV. We concentrated mainly on $Weyl^4$ and four derivative correction. While our results for $Weyl^4$ correction are in agreement with results

of [32], the four-derivative corrections are new results of this paper. We also compute the effect of Gauss-Bonnet to τ_π and κ . From our results we obtain the bound on Gauss-Bonnet coupling constant. In section V we consider R charge black hole in bulk and find the second order transport coefficients in presence of finite chemical potential. We also find the higher derivative effects on transport coefficients of field theory plasma dual to gauge supergravity theory. Finally, we analyze the flow equation of Green's function for boundary R current in section VI. Appendix A and B compliments some discussion on boundary terms. In appendix C, D and E, we have provided some long expressions.

II. FLOW OF RETARDED GREEN'S FUNCTION OF ENERGY-MOMENTUM TENSOR

In this section we briefly review the work of Liu and Iqbal [27]. They considered leading Einstein-Hilbert (E-H) action with a negative cosmological constant in 4+1 dimensions and studied the motion of a transverse graviton in this background⁶. The action is,

$$S_{EM} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} (R + 12). \quad (II.1)$$

The background has a black-brane solution as,

$$\begin{aligned} dS^2 &= g_{tt} dt^2 + g_{rr} dr^2 + g_{ij} dx^i dx^j, \\ g_{tt} &= -\frac{1-r^2}{r}, \quad g_{rr} = \frac{1}{4r^2(1-r^2)} \\ g_{ij} &= \frac{1}{r} \delta_{ij}. \end{aligned} \quad (II.2)$$

The solution is asymptotically AdS and it has a boundary topology $R \times R^3$. The horizon of the space-time is at $r \rightarrow 1$ and asymptotic boundary is at $r \rightarrow 0$.

We study the graviton's fluctuation in this background,

$$g_{xy} = g_{xy}^{(0)} + h_{xy}(r, x) = g_{xy}^{(0)} [1 + \epsilon \Phi(r, x)]. \quad (II.3)$$

By plugging it in the action and keeping terms to order ϵ^2 , we obtain the following effective action for the perturbation

$$S = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p,q=0}^2 \mathcal{A}_{p,q}(r, k) \phi^{(p)}(r, -k) \phi^{(q)}(r, k). \quad (II.4)$$

Here we use Fourier transform to work in the momentum space $k = \{-\omega, \vec{k}\}$,

$$\Phi(r, x) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \phi(r, k) \quad (II.5)$$

⁵ From Weyl invariance one can show that there are other transport coefficients in second order hydrodynamics [10]. However from the expansion of retarded Green's function it is only possible to compute only two of them.

⁶ Here we restrict ourselves to five space-time dimensions, but the discussions are quite generic and can be extended to arbitrary dimensions.

and $\phi^{(p)}(r, k)$ denotes the p^{th} derivative of the field $\phi(r, k)$ with respect to r ($p + q \leq 2$).

Next, we integrate by parts to obtain the bulk action in the following form (up to some total derivative terms)

$$S = \int \frac{d^4 k}{(2\pi)^4} dr (\mathcal{A}_1(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0(r, k) \phi(r, k) \phi(r, -k)), \quad (\text{II.6})$$

where,

$$\begin{aligned} \mathcal{A}_1(r, k) &= -\frac{\frac{1}{2} g^{rr} \sqrt{-g}}{16\pi G_5}, \\ \mathcal{A}_0(r, k) &= -\frac{\frac{1}{2} \sqrt{-g} g^{\mu\nu} k_\mu k_\nu}{16\pi G_5}. \end{aligned} \quad (\text{II.7})$$

From this action, we can find the conjugate momentum $\Pi(r, k_\mu)$ of the transverse graviton (for r-foliation) and the equation of motion,

$$\Pi(r, k_\mu) = 2\mathcal{A}_1(r, k) \phi'(r, k) \quad (\text{II.8})$$

and

$$\Pi'(r, k_\mu) - 2\mathcal{A}_0(r, k) \phi(r, k) = 0. \quad (\text{II.9})$$

The on-shell action reduces to the following surface term⁷,

$$S = \sum_{r=0,1} \int \frac{d^4 k}{(2\pi)^4} (\mathcal{A}_1(r, k) \phi'(r, k) \phi(r, -k)). \quad (\text{II.10})$$

Following the AdS/CFT prescription given in [5], the boundary retarded Green's function is given as,

$$G_R(k_\mu) = \lim_{r \rightarrow 0} \frac{2\mathcal{A}_1(r, k) \phi'(r, k) \phi(r, -k)}{\phi_0(k) \phi_0(-k)}, \quad (\text{II.11})$$

where, $\phi_0(k_\mu)$ is the value of the graviton fluctuation at boundary. Full solution of the graviton can be written as $\phi(r, k_\mu) = \phi_0(k_\mu) F(r, k_\mu)$, where $F(r, k_\mu)$ goes to identity at the boundary. We can rewrite the boundary retarded Green's function as,

$$G_R(k_\mu) = \lim_{r \rightarrow 0} \frac{\Pi(r, k_\mu)}{\phi(r, k_\mu)}. \quad (\text{II.12})$$

Let us define a response function of the boundary theory as⁸,

$$\bar{\chi}(k_\mu, r) = \frac{\Pi(r, k_\mu)}{i\omega \phi(r, k_\mu)} \quad (\text{II.13})$$

where $\omega = k_0$. This function is defined for all r and k_μ . Therefor the boundary Green's function is given by,

$$G_R(k_\mu) = \lim_{r \rightarrow 0} i\omega \bar{\chi}(k_\mu, r). \quad (\text{II.14})$$

We will study the radial evolution of the response function $\bar{\chi}(k_\mu)$ from horizon to boundary. Differentiating equation (II.13) and using the equations of motion (II.9) we get,

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}(k_\mu, r)^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right], \quad (\text{II.15})$$

where we define

$$\Sigma(r) = -2\mathcal{A}_1(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}} \quad (\text{II.16})$$

$$\Upsilon(r) = 2\mathcal{A}_0(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \quad (\text{II.17})$$

Putting values of \mathcal{A}_1 and \mathcal{A}_0 given in (II.7) we can easily recover the flow equation given in [27]⁹. However for future requirements, here we present it directly in terms of the coefficients of the graviton action.

As mentioned earlier, the flow equation in (II.15) is valid for any value of momentum. This is a first order differential equation and we need to specify one boundary condition to solve this equation. That naturally comes from the behavior of the equation at the horizon. Demanding the solution to be regular at the horizon, we get the following condition,

$$\bar{\chi}(k_\mu, r)^2 \Big|_{r=1} = \frac{\Sigma(r) \Upsilon(r)}{\omega^2} \Big|_{r=1}. \quad (\text{II.18})$$

For two derivative gravity this boundary condition implies that¹⁰,

$$\bar{\chi}(k_\mu, 1) = -\sqrt{\frac{\Sigma(1) \Upsilon(1)}{\omega^2}} = -\frac{1}{16\pi G_5} \quad (\text{II.19})$$

which is independent of k_μ . Therefore the full momentum response at the horizon corresponds to only to the zero momentum limit of boundary response, $\bar{\chi}(k_\mu, 1) = \bar{\chi}(k_\mu \rightarrow 0, r \rightarrow 0)$ ¹¹.

With this boundary condition, one can integrate out the differential equation (II.15) from horizon to asymptotic boundary and obtain the AdS/CFT response for all

⁷ We will discuss about other boundary terms in appendix B

⁸ We set the zero frequency part of G to zero, as it gives contact terms

⁹ just notice that g_{tt} there is negative of what we used here.

¹⁰ We choose the negative branch. The sign of the boundary condition II.18 depends on the choice of coordinate. In our coordinate the boundary is at $r \rightarrow 0$ hence we need to choose the negative branch.

¹¹ However, in higher derivative gravity we will see that the $\bar{\chi}(k_\mu, 1)$ depends on spatial momentum.

momentum k_μ . In particular, it is trivial to see that at $(\omega, k_i) \rightarrow 0$ limit, the flow is trivial

$$\partial_r \bar{\chi}(k_\mu, r) = 0 \quad (\text{II.20})$$

and using the boundary condition (II.18) we get the first order transport coefficient of boundary fluid, i.e. the shear viscosity coefficient turns out to be $\eta = \frac{1}{16\pi G_5}$.

In this paper, we will go away from $(\omega, k_i \rightarrow 0)$ limit. As we have already mentioned, it is possible to integrate the flow equation for any momentum (perturbatively) and we can easily find the higher order transport coefficients. The usual Kubo approach to compute these coefficients requires the full profile of the transverse graviton in black hole background (solving a second order differential equation), where as, using the flow equation, one can get these transport coefficients without explicit knowledge of the graviton's profile.

A. A renormalized response function

When we solve the flow equation (II.15) to get the boundary response function in general it involves divergence at the boundary ($r \rightarrow 0$). These are usual UV divergences and to remove them we need to re-normalize the response function properly.

We follow the holographic renormalization prescription of [34, 35]. As the graviton is massless, we only need to add the following counterterm to the graviton's action,

$$S_C = \frac{1}{16\pi G_5} \int_{r=\delta} d^4x \sqrt{-\gamma} \frac{1}{4} \Phi(\epsilon, x) \square \Phi(\epsilon, x). \quad (\text{II.21})$$

In momentum space,

$$S_C = \frac{1}{64\pi G_5} \int_{r=\delta} \frac{d^4k}{(2\pi)^4} \sqrt{-\gamma} \phi(\delta, k) (g^{tt}\omega^2 + k_i k^i) \phi(\delta, -k). \quad (\text{II.22})$$

Therefore the renormalized Green's function is given by,

$$\mathcal{G}_R = \lim_{r \rightarrow 0} \left[\frac{\Pi(r, k_\mu)}{\phi(r, k_\mu)} + \frac{\sqrt{-\gamma}}{32\pi G_5} (g^{tt}\omega^2 + k_i k^i) \right]. \quad (\text{II.23})$$

However we will study the flow of un-renormalized response function defined in (II.13) and we define our renormalized response function as,

$$\bar{\chi}^{Ren}(r, k_\mu) = \bar{\chi}(r, k_\mu) + \frac{1}{i\omega} \frac{\sqrt{-\gamma} (g^{tt}\omega^2 + k_i k^i)}{32\pi G_5}. \quad (\text{II.24})$$

The counter term will cancel the UV divergences appearing in the expression of $\bar{\chi}$ and we will get a finite result at the boundary, i.e. $\lim_{r \rightarrow 0} \bar{\chi}^{Ren}(r, k_\mu)$ will be finite. From the above analysis, we understand that one can get rid of the UV divergences appearing in the response function by following the holographic renormalization technique. But, an important observation is, this counter term does not add any finite contribution to the result

it only cancels out the divergences. Thus, one can study the flow of the un-renormalized response function and ignore the divergences piece to get the finite contribution at the boundary.

B. Second order transport coefficients from flow equation

In this subsection, we compute the higher order transport coefficients by solving the flow equation (II.15) perturbatively up to order ω^2 and k_i^2 . This is a non-linear first order differential equation. Now, the right hand side of this equation is proportional to ω . Hence, to solve $\bar{\chi}$ to order ω^2 , we can replace the leading order solution for $\bar{\chi}$ in the right hand side of equation (II.15). This simplifies the situation a lot as the non-linear equation becomes linear. Now, to leading order, $\bar{\chi} = -\eta = -\frac{1}{16\pi G_5}$. Therefore up to order ω^2 , we get,

$$\partial_r \bar{\chi}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\eta^2}{\Sigma(r)} - \frac{\Upsilon(r)}{\omega^2} \right] + \mathcal{O}(\omega^2, k_i^2). \quad (\text{II.25})$$

The integration constant for the equation can be fixed from the boundary condition (II.18). Putting the value of the constant, the solution takes the form,

$$\begin{aligned} i\omega \bar{\chi}(k_\mu, 0) &= \lim_{r \rightarrow 0} -\frac{1}{96\pi G_5 r} \left[3q^2(r-1) + \omega(3\omega \right. \\ &\quad \left. + r(\omega(\log(8) - 3) + 6i)) \right] \\ &\quad + \mathcal{O}(q\omega^2, \omega q^2, q^3, \omega^3) \\ &= -i\omega \left(\frac{1}{16\pi G_5} \right) \\ &\quad + \omega^2 \left[\frac{1}{2}(1 - \ln 2) \left(\frac{1}{16\pi G_5} \right) \right] \\ &\quad - \frac{q^2}{2} \left(\frac{1}{16\pi G_5} \right) + \mathcal{O}\left(\frac{1}{r}\right). \end{aligned} \quad (\text{II.26})$$

Here we have chosen the four momentum in the following form

$$k = \{\omega, 0, 0, q\}. \quad (\text{II.27})$$

This expression has divergence as $r \rightarrow 0$ (UV divergence) and can be removed by adding suitable counter term (as explained in the last section).

Comparing the finite piece of (II.26) at $r \rightarrow 0$ with the generic expansion of the retarded Green's function (I.2)¹², we get,

$$\begin{aligned} \eta &= \frac{T^3 \pi^3}{16\pi G_5}, \\ \tau_\pi &= \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T}. \end{aligned} \quad (\text{II.28})$$

¹² The overall sign depends on the choice of coordinate.

Here $T = \frac{1}{\pi}$. These results are in agreement with [10]. In appendix A we briefly outline the Kubo method to get this result. Thus, we see that, studying the flow equation of the response function we can compute the higher order transport coefficients perturbatively. Here, we present the results for the second order transport coefficients, but, in general it is possible to go beyond second order.

At this point, it is not clear why only considering the boundary term from action (II.6) is enough to get the correct results. In usual Kubo approach, one needs to take into account the Gibbons-Hawking term also. But as the action (II.6) has well defined variational principle, one does not need to add any Gibbons-Hawking term with it. In appendix B we show that the boundary terms coming from the original action and the corresponding Gibbons-Hawking action are exactly same as the boundary terms coming from the action (II.6) up to terms proportional to ϕ^2 and pure divergence terms. The ϕ^2 terms do not contribute to any transport coefficients¹³. The divergent terms will get canceled by the proper counterterms and hence are not important for finding the transport coefficients. Thus it is clear that the effective action will give us the correct transport coefficients for the boundary plasma. This observation also holds for higher derivative gravity theory¹⁴.

III. HIGHER DERIVATIVE CORRECTION TO FLOW EQUATION

So far we have discussed the flow equation of two point correlation function of energy-momentum tensor of boundary theory whose gravity dual is given by Einstein-Hilbert action (two derivative action). But it is not obvious how to generalize this for higher derivative case. The proof given in [27] was based on the canonical form (II.6) of graviton's action. In presence of arbitrary higher derivative terms in the bulk, the general action for the perturbation h_{xy} does not have the above form (II.6). Rather it will have more than two derivative (with respect to r) terms like $\phi'\phi''$, ϕ''^2 etc. In presence of these terms it is not possible to bring this action into a canonical form (up to some total derivative terms). In this paper we consider generic higher derivatives terms in the bulk Lagrangian. We follow the prescription of [29] to construct an effective action " S_{eff} " for transverse graviton in canonical form in presence of generic higher derivative terms in the bulk. The effective action and original action give same equation of motion perturbatively in the coupling of the higher derivative terms.

Let us consider a gravity set-up with n derivative ac-

tion.

$$\mathcal{I} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R + 12 + \alpha' R^{(n)} \right] \quad (\text{III.1})$$

where, $\mathcal{R}^{(n)}$ is any n derivative Lagrangian. The metric in general is given by (assuming planar symmetry),

$$ds^2 = -(h_t(r) + \alpha' h_t^{(n)}(r))dt^2 + \frac{dr^2}{h_r(r) + \alpha' h_r^{(n)}(r)} + \frac{1}{r}(1 + \alpha' h_s^{(n)}(r))d\vec{x}^2. \quad (\text{III.2})$$

Substituting the background metric with fluctuation (II.3) in action (III.1) (we call it general action or original action) for the scalar field $\phi(r, k)$ we get,

$$\mathcal{I} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p,q=0}^n \mathcal{A}_{p,q}(r, k) \phi^{(p)}(r, -k) \phi^{(q)}(r, k) \quad (\text{III.3})$$

where, $\phi^{(p)}(r, k)$ denotes the p^{th} derivative of the field $\phi(r, k)$ with respect to r and $p + q \leq n$. The coefficients $\mathcal{A}_{p,q}(r, k)$ in general depends on the coupling constant α' . $\mathcal{A}_{p,q}$ with $p + q \geq 3$ are proportional to α' and vanishes in $\alpha' \rightarrow 0$ limit, since the terms $\phi^{(p)}\phi^{(q)}$ with $p + q \geq 3$ appears as an effect of higher derivative terms in the action (III.1).

Up to some total derivative terms, the general action (II.6) can also be written as,

$$\begin{aligned} \mathcal{I} &= \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p=0}^{n/2} \mathcal{A}_p(r, k) \phi^{(p)}(r, -k) \phi^{(p)}(r, k) \\ &\quad (\text{for } n \text{ even}) \\ &= \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \sum_{p=0}^{\frac{n-1}{2}} \mathcal{A}_p(r, k) \phi^{(p)}(r, -k) \phi^{(p)}(r, k) \\ &\quad (\text{for } n \text{ odd}). \end{aligned} \quad (\text{III.4})$$

However this action does not have canonical form. We write an effective action for transverse graviton in canonical form,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[\mathcal{A}_1^{\text{HD}}(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0^{\text{HD}}(r, k) \phi(r, k) \phi(r, -k) \right] \quad (\text{III.5})$$

with some unknown function $\mathcal{A}_1^{\text{HD}}$ and $\mathcal{A}_0^{\text{HD}}$. We fix these functions by demanding that the equations of motion obtained from the effective action and the original action are same perturbatively in α' .

The generalized canonical momentum and equation of motion are given by,

$$\begin{aligned} \Pi^{\text{HD}}(r, k) &= 2\mathcal{A}_1^{\text{HD}}(r, k) \phi'(r, k) \\ (\Pi^{\text{HD}}(r, k))' &= 2\mathcal{A}_0^{\text{HD}}(r, k) \phi(r, k). \end{aligned} \quad (\text{III.6})$$

¹³ They only contribute to pressure of the boundary theory.

¹⁴ In appendix B, we have proved this explicitly for $R^{(n)}$ gravity theory.

Once we find the effective action for the graviton, we follow the procedure in the previous section to obtain the flow equation for the boundary Green's function in generic higher derivative gravity.

The boundary Green's function is given by,

$$G_R^{\text{HD}}(k_\mu) = \lim_{r \rightarrow 0} \frac{2\mathcal{A}_1^{\text{HD}}(r, k)\phi'(r, k)\phi(r, -k)}{\phi_0(k)\phi_0(-k)}, \quad (\text{III.7})$$

which can be written using the definition of canonical momentum as,

$$G_R^{\text{HD}}(k_\mu) = \lim_{r \rightarrow 0} \frac{\Pi^{\text{HD}}(r, k_\mu)}{\phi(r, k_\mu)}. \quad (\text{III.8})$$

Let us define a response function of the boundary theory in higher derivative theory as,

$$\bar{\chi}^{\text{HD}}(k_\mu, r) = \frac{\Pi^{\text{HD}}(r, k_\mu)}{i\omega\phi(r, k_\mu)}. \quad (\text{III.9})$$

Therefore the flow equation is given by,

$$\partial_r \bar{\chi}^{\text{HD}}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}^{\text{HD}}(k_\mu, r)^2}{\Sigma^{\text{HD}}(r, k)} - \frac{\Upsilon^{\text{HD}}(r, k)}{\omega^2} \right], \quad (\text{III.10})$$

where we define

$$\Sigma^{\text{HD}}(r, k) = -2\mathcal{A}_1^{\text{HD}}(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}} \quad (\text{III.11})$$

$$\Upsilon^{\text{HD}}(r, k) = 2\mathcal{A}_0^{\text{HD}}(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \quad (\text{III.12})$$

This is the flow equation for two point correlation function of energy-momentum tensor in presence of generic higher derivative term in the bulk action. Therefore integrating this equation from horizon to asymptotic boundary one can find the higher derivative correction to the transport coefficients at any order in frequency/momentum.

Like two derivative case here also we need to provide a boundary condition to solve this equation. The response function $\bar{\chi}^{\text{HD}}(k_\mu, r)$ should be well-defined at horizon. This implies,

$$\bar{\chi}^{\text{HD}}(k_\mu, r) \Big|_{r=r_h} = \sqrt{\frac{\Sigma^{\text{HD}}(r)\Upsilon^{\text{HD}}(r)}{\omega^2}} \Big|_{r=r_h} \quad (\text{III.13})$$

here the horizon is located at $r = r_h$.

One important point to mention here is that unlike two derivative gravity where $\bar{\chi}(k_\mu, r_h)$ was independent of k_μ , $\bar{\chi}^{\text{HD}}(k_\mu, r_h)$ can in general depend on k_μ . We will see this explicitly in the next section. Therefore the full momentum response at the horizon may not be able to correspond only to the zero momentum limit of boundary response in higher derivative theory.

Like two derivative case, the response function in higher-derivative gravity theory also contains UV divergences. We need to add proper counter term following the

holographic renormalization procedure to cancel these divergences. A little more thinking also says that in presence of any higher-derivative term in the action the structure of the counterterm remains same as (II.21). Only the overall normalization constant depends on higher-derivative coupling. Thus, similar to the leading gravity, the counterterm in higher derivative gravity also cancels out the divergence and does not add any finite contribution to the boundary response function. One can study the flow equation of the un-renormalized response function and read off the transport coefficients from its finite piece.

IV. EXAMPLES: STRING THEORY CORRECTIONS TO FLOW EQUATION

String theory predicts next to leading order corrections to Einstein-Hilbert action. These corrections are relevant at a distance comparable with typical length scale of the theory $l_s = \sqrt{\alpha'}$. The short distance corrections to this action is described by supplementing this action by higher curvature terms. However, here we treat the stringy effects perturbatively *i.e* the coupling of higher derivative terms to be small.

From the point of view of AdS/CFT the small α' correction in supergravity corresponds to $\frac{1}{\lambda}$ correction in strongly coupled gauge theory in planar limit where λ is the 't Hooft coupling. The precise dictionary between string length and 't Hooft coupling is,

$$\alpha'^2 = \frac{L^4}{4\pi\lambda}, \quad L \text{ AdS radius.} \quad (\text{IV.1})$$

In this section, we will consider two examples of higher-derivative terms coming from string theory and study their effects on flow equations and second order transport coefficients.

A. Weyl⁴ term

We consider the well known Weyl⁴ term. This term appears in type II string theory. Adding this term in the bulk action corresponds to $\frac{1}{\lambda^{3/2}}$ correction in dual large N theory. The string theory correction to second order transport coefficients have already been computed in [32] using usual *Kubo* formula. Here, we show that one can obtain the correct result by studying the *first order* flow equation in higher derivative gravity without solving any second order differential equation for graviton.

The five dimensional bulk action is given by,

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + 12 + \gamma W^{(4)} \right) \quad (\text{IV.2})$$

where, the coupling constant γ is given by,

$$\gamma = \frac{1}{8} \zeta(3) \alpha'^3 \quad (\text{IV.3})$$

and

$$W^{(4)} = C^{hmnk} C_{pmnq} C_h^{rsp} C_{rsk}^q + \frac{1}{2} C^{hkmn} C_{pqmn} C_h^{rsp} C_{rsk}^q \quad (\text{IV.4})$$

and the Weyl tensors C_{abcd} are given by,

$$C_{abcd} = R_{abcd} + \frac{1}{3} (g_{ad} R_{cb} + g_{bc} R_{ad} - g_{ac} R_{db} - g_{bd} R_{ca}) + \frac{1}{12} (g_{ac} g_{bd} - g_{ad} g_{cb}) R. \quad (\text{IV.5})$$

The background metric is given by [36, 37]¹⁵,

$$ds^2 = - \left(\frac{(1-r^2)r_0^2}{r} - 15r(3r^6 - 8r^4 + 5)\gamma r_0^2 \right) dt^2 + \left(\frac{1}{4r^2 - 4r^4} + \frac{(-285r^4 + 75r^2 + 75)\gamma}{4 - 4r^2} \right) dr^2 + \frac{1}{r} d\vec{x}^2. \quad (\text{IV.6})$$

The temperature of this black hole is given by,

$$T = \frac{r_0}{\pi} (1 + 15\gamma). \quad (\text{IV.7})$$

and the horizon is located at $r = 1$.

The effective action for transverse graviton has been computed in [29]. Here we will present the result only. The effective action is given by,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} dr \left[\mathcal{A}_1^{W^4}(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0^{W^4}(r, k) \phi(r, k) \phi(r, -k) \right] \quad (\text{IV.8})$$

where, $\mathcal{A}_1^{W^4}$ and $\mathcal{A}_0^{W^4}$ are given in appendix C.

Therefore the flow equation is given by,

$$\partial_r \bar{\chi}^{W^4}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}^{W^4}(k_\mu, r)^2}{\Sigma^{W^4}(r, k)} - \frac{\Upsilon^{W^4}(r, k)}{\omega^2} \right], \quad (\text{IV.9})$$

where,

$$\Sigma^{W^4}(r, k) = -2\mathcal{A}_1^{W^4}(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}} \quad (\text{IV.10})$$

$$\Upsilon^{W^4}(r, k) = 2\mathcal{A}_0^{W^4}(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \quad (\text{IV.11})$$

The explicit expressions for Σ and Υ can be obtained by using $\mathcal{A}_1^{W^4}$ and $\mathcal{A}_0^{W^4}$.

From the regularity of $\bar{\chi}^{W^4}(k_\mu, r)$ at horizon we get,

$$\bar{\chi}^{W^4}(k_\mu, 1) = \sqrt{\frac{\Sigma^{W^4}(1) \Upsilon^{W^4}(1)}{\omega^2}} = \frac{r_0^3}{16\pi G_5} + \frac{\gamma r_0}{4\pi G_5} (45r_0^2 + 11q^2). \quad (\text{IV.12})$$

Here, we see that unlike the two-derivative gravity, the horizon value of the response function depends on spatial momenta q (see Fig. 2). With this boundary condition we solve the flow equation up to order ω^2 and q^2 (ignoring $\mathcal{O}(\omega q^2)$ term). Here we write the final result¹⁶.

$$i\omega \bar{\chi}^{W^4}(k_\mu, 0) = -i(1 + 180\gamma) \frac{r_0^3}{16\pi G_5} \omega + \left[\frac{1}{2} (1 - \log(2)) + \frac{5}{4} \gamma (199 - 66 \log(2)) \right] \frac{r_0^2}{16\pi G_5} \omega^2 - \frac{1}{2} (1 + 20\gamma) \frac{r_0^2}{16\pi G_5} q^2 + \mathcal{O}(q\omega^2, \omega q^2, q^3, \omega^3). \quad (\text{IV.13})$$

Comparing this result with (I.2) we get

$$\frac{\eta}{\pi^3 T^3} = 1 + 135\gamma + \mathcal{O}(\gamma^2) \\ \kappa = \frac{\eta}{\pi T} (1 - 145\gamma) + \mathcal{O}(\gamma^2) \\ \tau_\pi T = \frac{2 - \log(2)}{2\pi} + \frac{375\gamma}{4\pi} + \mathcal{O}(\gamma^2). \quad (\text{IV.14})$$

These results are in agreement with [32]. It provides a non-trivial check to this approach of obtaining higher order transport coefficients from the flow equation (II.15).

B. Four derivative term

In this section, we will concentrate on the generic four derivative corrections to Einstein-Hilbert action. These terms arise in the effective action for the heterotic string theory. In fact, the complete super-symmetrized R^2 correction to effective Heterotic string theory is known and one way to obtain it is the super-symmetrization of the Lorentz Chern-Simons terms [38, 39]. This terms also arises in the context of Type IIB string theory [40, 41], where the theory is on $AdS_5 \times X^5$, the compact space X^5 being S^5/Z_2 . The dual theory is $\mathcal{N} = 2Sp(N)$ gauge theory with 4 fundamental and 1 antisymmetric traceless hyper-multiplets. This super-conformal theories arises in the context of ND3-branes sitting inside 8 D7-branes co-incident on an orientifold 7-plane. In this case, generic

¹⁵ In this particular example we keep the extremality parameter r_0 explicitly. It would help us to write the relation between different transport coefficients. One can set r_0 to be one by time re-scaling what we have done in the next example.

¹⁶ $k = \{\omega, 0, 0, q\}$ and we ignore the UV divergence piece.

four derivative R^2 correction comes from the DBI action of the branes.

Here we compute the generic four derivative correction to the second order transport coefficients, the relaxation time τ_π and κ . We can choose the coefficients of the higher derivative terms to be the four-dimensional Euler density and get pure Gauss-Bonnet correction to these coefficients.

The action

$$\mathcal{I} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R + 12 + \alpha' \left(\beta_1 R^2 + \beta_2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \beta_3 R_{\mu\nu} R^{\mu\nu} \right) \right]. \quad (\text{IV.15})$$

In particular for Gauss-Bonnet correction, $\beta_1 = 1, \beta_2 = 1, \beta_3 = -4$. One can get rid of the *Ricci*² and *Scalar*² terms by a field redefinition and therefore all physical quantities should depend on the coefficient β_2 only. Here, we prefer to work with the generic case as it would be easier for us to get the results for pure Gauss-Bonnet combination at every step.

The background solution is given by [30],

$$ds^2 = f(r)dt^2 + \frac{g(r)}{4r^3}dr^2 + \frac{1}{r}d\vec{x}^2 \quad (\text{IV.16})$$

where $f(r)$ and $g(r)$ are given by,

$$f(r) = r - \frac{1}{r} - 2r(r^2 - 1)\beta_2\alpha' \quad (\text{IV.17})$$

and

$$g(r) = \frac{r}{1-r^2} + \frac{2r(10\beta_1 + (1-3r^2)\beta_2 + 2\beta_3)\alpha'}{3(r^2-1)}. \quad (\text{IV.18})$$

This is the background metric corrected up to order α' . We have fixed the integration constant such that the boundary metric is Minkowskian and the horizon is located at $r = 1$. The temperature of the black brane is given by,

$$T = \frac{1}{\pi} + \frac{10\beta_1 - 5\beta_2 + 2\beta_3}{3\pi}\alpha'. \quad (\text{IV.19})$$

Similar to the *Weyl*⁴ case, we can write the following effective action for this model,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} dr \left[\mathcal{A}_1^{GB}(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0^{GB}(r, k) \phi(r, k) \phi(r, -k) \right] \quad (\text{IV.20})$$

where, \mathcal{A}_1^{GB} and \mathcal{A}_0^{GB} are given in appendix D. Now, it is straightforward to write the corresponding flow equation (II.15) in this case,

$$\partial_r \bar{\chi}^{GB}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}^{GB}(k_\mu, r)^2}{\Sigma^{GB}(r, k)} - \frac{\Upsilon^{GB}(r, k)}{\omega^2} \right], \quad (\text{IV.21})$$

where we define

$$\Sigma^{GB}(r, k) = -2\mathcal{A}_1^{GB}(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}} \quad (\text{IV.22})$$

$$\Upsilon^{GB}(r, k) = 2\mathcal{A}_0^{GB}(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \quad (\text{IV.23})$$

Now, the boundary condition (II.18) takes the following form,

$$\bar{\chi}^{GB}(k_\mu, 1) = \frac{1}{16\pi G_5} [1 + ((q^2 - 8)\beta_3 - 40\beta_1)\alpha'] . \quad (\text{IV.24})$$

As mentioned earlier, we see that even in this case, the boundary condition depends on spatial momenta q through the coefficient β_3 . With this boundary condition, one can solve the flow equation (IV.21) and the solution is given by,

$$\begin{aligned} i\omega \bar{\chi}^{GB}(k_\mu, 0) = & \frac{1}{16\pi G_5} \left[-i(1 - (40\beta_1 + 8\beta_3)\alpha')\omega \right. \\ & + \frac{\omega^2}{2} \left[(1 - \log 2) + \frac{\alpha'}{6} (130\beta_1(\log 2 - 1) \right. \\ & \left. \left. - \beta_2(5\log 2 - 2) + 26\beta_3(\log 2 - 1)) \right] \right. \\ & \left. - \frac{q^2}{2} \left[1 - \frac{1}{3} (130\beta_1 + 25\beta_2 + 26\beta_3)\alpha' \right] \right] \\ & + \mathcal{O}(q\omega^2, \omega q^2, q^3, \omega^3) . \end{aligned} \quad (\text{IV.25})$$

From this expression we get the following transport coefficients,

$$\eta = \frac{1}{16\pi G_5} (1 - 8(5\beta_1 + \beta_3)\alpha') + \mathcal{O}(\alpha'^2). \quad (\text{IV.26})$$

This matches with results in [40, 42, 43]. The higher order coefficients are,

$$\begin{aligned} \kappa &= \frac{\eta}{\pi T} (1 - 10\beta_2\alpha') + \mathcal{O}(\alpha'^2) \\ \tau_\pi T &= \frac{2 - \ln 2}{2\pi} - \frac{11\beta_2}{2\pi}\alpha' + \mathcal{O}(\alpha'^2). \end{aligned} \quad (\text{IV.27})$$

As we can see, the physical quantities $\eta/s, \kappa, \tau_\pi T$ only depend on the coefficient β_2 . In particular to Gauss-Bonnet combination, the corrections are,

$$\begin{aligned} \kappa &= \frac{\eta}{\pi T} (1 - 10\alpha') + \mathcal{O}(\alpha'^2) \\ \tau_\pi T &= \frac{2 - \ln 2}{2\pi} - \frac{11}{2\pi}\alpha' + \mathcal{O}(\alpha'^2). \end{aligned} \quad (\text{IV.28})$$

(IV.27) and (IV.28) are new results of this paper.

1. Exact result for Gauss-Bonnet black hole

As we have done the above computation perturbatively, the above expressions are valid only at order α' . But, one can consider the Gauss-Bonnet term exactly in

coupling. For pure Gauss-Bonnet combination the equations of motion remain second order differential equation and hence it is easy to solve exactly to find the background space-time. We solve the flow equation in this background exactly in coupling constant, and find the exact expressions for relaxation time τ_π and κ . In this section we briefly outline the result.

The action and the solution is given by,

$$\begin{aligned} \mathcal{I}_{GB} = & \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R + 12 \right. \\ & \left. + \frac{\lambda_{gb}}{2} \left(R^2 + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} \right) \right] \\ ds^2 = & r^2 \left(-\frac{f(r)}{f_\infty} dt^2 + d\vec{x}^2 \right) + \frac{dr^2}{r^2 f(r)} \end{aligned} \quad (\text{IV.29})$$

where,

$$f_\pm = \frac{1}{2\lambda_{gb}} \left[1 \pm \sqrt{1 - 4\lambda_{gb} \left(1 - \frac{r_0^4}{r^4} \right)} \right] \quad (\text{IV.30})$$

and

$$f_\infty = \lim_{r \rightarrow \infty} f(r) = \frac{1 - \sqrt{1 - 4\lambda_{gb}}}{2\lambda_{gb}}. \quad (\text{IV.31})$$

In this coordinate the boundary metric is η . We also consider only the $'-'$ branch of f_\pm which corresponds to a non-singular black hole solution with non-degenerate horizon.

The black hole temperature is given by,

$$T = \frac{r_0}{\pi f_\infty}. \quad (\text{IV.32})$$

With exact GB, the effective action for fluctuation has a canonical form. Therefore we derive the flow equation for the response function (as we did in section II) and solving this equation we get,

$$\begin{aligned} \eta = & \frac{1}{16\pi G_5} (1 - 4\lambda_{gb}) \\ \kappa = & \frac{2\lambda_{gb} (8\lambda_{gb} - 1)}{(1 - \sqrt{1 - 4\lambda_{gb}}) (4\lambda_{gb} - 1)} \\ \tau_\pi T = & \frac{1}{4\pi(-1 + 4\lambda_{gb})} \left[-8\lambda_{gb}^2 + 12\sqrt{1 - 4\lambda_{gb}}\lambda_{gb} \right. \\ & + 10\lambda_{gb} - 2\sqrt{1 - 4\lambda_{gb}} - 4\log(2)\lambda_{gb} \\ & + (1 - 4\lambda_{gb}) \log \left(-4\lambda_{gb} + \sqrt{1 - 4\lambda_{gb}} + 1 \right) \\ & \left. + (4\lambda_{gb} - 1) \log (1 - 4\lambda_{gb}) - 2 + \log(2) \right]. \end{aligned} \quad (\text{IV.33})$$

One can easily check that up to first order in λ_{gb} , the results in (IV.33) reduces to the one in (IV.28). In [44] the authors obtained the relation between second order transport coefficients and λ_{gb} numerically, however we are able to present the result exactly.

As we have mentioned in introduction that the flow equation is a first order non-linear differential equation but one can reduce this equation to a second order linear differential equation. This second order differential equation is related to the equation of motion for transverse graviton (gauge invariant excitations). Therefore we can use this equation to study causality violation in Gauss-Bonnet gravity. In [44, 45] it was found that to preserve causality of a conformal fluid there exists a bound on second order transport coefficients,

$$\tau_\pi T - 2\frac{\eta}{s} \geq 0. \quad (\text{IV.34})$$

In Fig 3 we plot $\tau_\pi T - 2\frac{\eta}{s}$ for our result and find the following bound on λ_{gb} which is in agreement with [44].

$$-0.711 \leq \lambda_{gb} \leq 0.113. \quad (\text{IV.35})$$

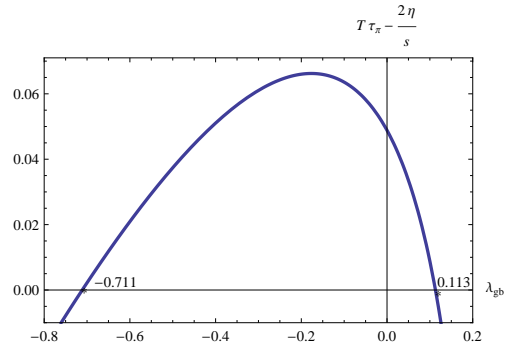


FIG. 3. Bound on λ_{gb} .

V. FLOW EQUATION FOR CHARGED BLACK HOLES

Electrically charged black holes in five dimensions have drawn a lot of interest in the context of AdS/CFT. The electric charge of these black holes are mapped to the global R-charge of the dual field theory. Because of the presence of the electric charges, the thermodynamics and the phase structure of these black holes are rather complicated and also interesting at the same time. There have been a lot of study of thermodynamics and phase transitions of these charged black hole with different horizon topologies (see [46] and references therein).

The goal of the present section is to apply the AdS/CFT correspondence to understand how non-vanishing chemical potentials effect the hydrodynamic behavior of strongly coupled gauge theories. We study the second order hydrodynamics in two cases: (a) Generic R-charge black holes and (b) Charge black holes in higher-derivative gravity.

A. R-charged black holes

We consider a conformal field theory with conserved charge (density) in addition to energy and momentum. This is especially an interesting extension of the hydrodynamics of the uncharged fluids.

The second order hydrodynamics of charged fluid has been studied in [47, 48]. They consider Reissner-Nordstrom black hole in five dimensions and found the effect of chemical potential on second order transport coefficients in some limits of chemical potential. However, we consider generic R -charged black holes with three (unequal) charges (chemical potentials) and find the exact expressions for second order transport coefficients in presence of three chemical potentials. As we have mentioned in the introduction that solving the flow equation (of retarded Green's function of energy momentum tensor) we can only find two second order transport coefficients whereas in [47, 48] all other second order transport coefficients have been reported.

We consider R -charged black holes in five dimensions. A consistent truncation of $\mathcal{N} = 8$, $D = 5$ gauged supergravity with $SO(6)$ Yang-Mills gauge group, which can be obtained by S^5 reduction of type IIB supergravity, gives rise to $\mathcal{N} = 2$, $D = 5$ gauge supergravity with $U(1)^3$ gauge group. The same theory can also be obtained by compactifying eleven dimensional supergravity, low energy theory of M theory, on a Calabi-Yau three folds. The bosonic part of the action of $\mathcal{N} = 2$, $D = 5$ gauged supergravity is given by [46]. We follow the notation of [49].

$$\begin{aligned} \mathcal{I}_{\text{sugra}} = & \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R + V(X) \right. \\ & \left. - \frac{1}{2} G_{IJ}(X) F_{\mu\nu}^I F^{\mu\nu J} - G_{IJ}(X) \partial_\mu X_I \partial^\mu X_J \right] \\ & + \frac{\xi}{16\pi G_5} \int d^5x \epsilon^{\mu\nu\rho\sigma\gamma} A_\mu F_{\nu\rho} F_{\sigma\gamma} \end{aligned} \quad (\text{V.1})$$

where, X^I 's are three real scalar fields, subject to the constraint $X^1 X^2 X^3 = 1$. F^I 's, which are field strengths of three Abelian gauge fields ($I, J=1, 2, 3$), and the scalar potential $V(X)$ is given by,

$$\begin{aligned} F_{\mu\nu}^I &= 2\partial_{[\mu} A_{\nu]}^I \\ G_{IJ} &= \frac{1}{2} \text{diag} [(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}] \\ V(X) &= 2 \sum_I \frac{1}{X_I} \end{aligned} \quad (\text{V.2})$$

The three-charge non-extremal STU solution is specified by the following background values of the metric

$$ds^2 = -\mathcal{H}^{-2/3} f_k dt^2 + \mathcal{H}^{1/3} (f_k^{-1} dr^2 + r^2 d\Omega_{3,k}^2), \quad (\text{V.3})$$

$$\begin{aligned} f_k &= k - \frac{m_k}{r^2} + r^2 \mathcal{H}, \quad H_i = 1 + \frac{q_i}{r^2}, \\ \mathcal{H} &= H_1 H_2 H_3, \end{aligned} \quad (\text{V.4})$$

as well as the scalar and the gauge fields

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i}, \quad A_t^i = \sqrt{\frac{k q_i + m_k}{q_i}} (1 - H_i^{-1}). \quad (\text{V.5})$$

The parameter k determines the spatial curvature of $d\Omega_{3,k}^2$: $k = 1$ corresponds to the metric on the three-sphere of unit radius, $k = 0$ - to the metric on R^3 . Hydrodynamic approximation is valid only in the case of a translational-invariant horizon, in our case we set $k = 0$ and

$$d\Omega_{3,0}^2 \rightarrow (dx^2 + dy^2 + dz^2).$$

Replacing the radial coordinate $r \rightarrow r_0/\sqrt{r}$, where r_0 is the largest root of the equation $f(r) = 0$, the background solution in this new coordinate is given by,

$$\begin{aligned} ds_5^2 = & -\mathcal{H}^{-2/3} \frac{(\pi T_0)^2}{r} f dt^2 + \mathcal{H}^{1/3} \frac{1}{4fr^2} dr^2 \\ & + \mathcal{H}^{1/3} \frac{(\pi T_0)^2}{r} (dx^2 + dy^2 + dz^2), \end{aligned} \quad (\text{V.6})$$

$$\begin{aligned} f(r) &= \mathcal{H}(r) - r^2 \prod_{i=1}^3 (1 + \kappa_i), \quad H_i = 1 + \kappa_i r, \\ \kappa_i &\equiv \frac{q_i}{r_0^2}. \end{aligned} \quad (\text{V.7})$$

where κ_i 's are chemical potentials and

$$T_0 = r_0/\pi. \quad (\text{V.8})$$

The scalar fields and the gauge fields are given by

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i(u)}, \quad A_t^i = \frac{\tilde{\kappa}_i \sqrt{2}u}{L H_i(u)} \quad (\text{V.9})$$

where,

$$\tilde{\kappa}_i = \sqrt{q_i} \prod_{i=1}^3 (1 + \kappa_i)^{1/2}. \quad (\text{V.10})$$

The Hawking temperature of the background (V.6) is given by

$$T_H = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3}{2\sqrt{(1 + \kappa_1)(1 + \kappa_2)(1 + \kappa_3)}} T_0. \quad (\text{V.11})$$

We perturb the xy component of background metric and the action for transverse graviton is given by,

$$\begin{aligned} S_{\text{eff}} = & \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} dr \left[\mathcal{A}_1^Q(r, k) \phi'(r, k) \phi'(r, -k) \right. \\ & \left. + \mathcal{A}_0^Q(r, k) \phi(r, k) \phi(r, -k) \right] \end{aligned} \quad (\text{V.12})$$

where,

$$\mathcal{A}_1^Q = -\frac{r_0^4 f(r)}{r} \quad (\text{V.13})$$

and

$$\mathcal{A}_0^Q = \frac{r_0^2}{4r^2} \left(\frac{H_1 H_2 H_3}{f(r)} - q^2 \right). \quad (\text{V.14})$$

Therefore the flow equation is given by,

$$\partial_r \bar{\chi}^Q(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}^Q(k_\mu, r)^2}{\Sigma^Q(r, k)} - \frac{\Upsilon^Q(r, k)}{\omega^2} \right]. \quad (\text{V.15})$$

Solving this equation perturbatively in ω and q we get

$$\begin{aligned} \bar{\chi}^Q(k_\mu, r) = & -\frac{r_0^3 \prod_i (1 + \kappa_i)^{1/2}}{16\pi G_5} \\ & + \frac{ir_0^2 (q^2 - \omega^2)}{2\omega} \frac{1}{16\pi G_5} \left(1 - \frac{1}{r} \right) \\ & + \frac{i\omega r_0^2 \prod_i (1 + \kappa_i)}{16\pi G_5 \sqrt{4P_\kappa + (1 + S_\kappa)^2}} \\ & \left(\frac{1}{2} \ln \left[\frac{1 + S_\kappa - 2P_\kappa - \sqrt{4P_\kappa + (1 + S_\kappa)^2}}{1 + S_\kappa - 2rS_\kappa - \sqrt{4P_\kappa + (1 + S_\kappa)^2}} \right] \right. \\ & \left. + \frac{1}{2} \ln \left[\frac{1 + S_\kappa - 2rP_\kappa + \sqrt{4P_\kappa + (1 + S_\kappa)^2}}{1 + S_\kappa - 2S_\kappa + \sqrt{4P_\kappa + (1 + S_\kappa)^2}} \right] \right) \\ & + \mathcal{O}(q\omega^2, \omega q^2, q^3, \omega^3) \end{aligned} \quad (\text{V.16})$$

where,

$$\begin{aligned} S_\kappa &= \sum_i \kappa_i \\ P_\kappa &= \prod_i \kappa_i. \end{aligned} \quad (\text{V.17})$$

Computing the response function at the boundary (throwing away the divergent piece) we get the following transport coefficients,

$$\eta = \frac{r_0^3}{16\pi G_5} \prod_i (1 + \kappa_i)^{1/2} \quad (\text{V.18})$$

and

$$\begin{aligned} \kappa &= \frac{\eta}{\pi T} \frac{1 + S_\kappa/2 - P_\kappa/2}{\prod_i (1 + \kappa_i)} \\ \tau_\pi T &= \frac{2 + S_\kappa - P_\kappa}{4\pi \prod_i (1 + \kappa_i)} \left[2 - \frac{\prod_i (1 + \kappa_i)}{\sqrt{4P_\kappa + (1 + S_\kappa)^2}} \right. \\ & \quad \left. \ln \left(\frac{3 + S_\kappa + \sqrt{4P_\kappa + (1 + S_\kappa)^2}}{3 + S_\kappa - \sqrt{4P_\kappa + (1 + S_\kappa)^2}} \right) \right]. \end{aligned} \quad (\text{V.19})$$

These are the new results in this paper. It is easy to check that for $\kappa_i \rightarrow 0$ limit we recover the results in section II.

To complete the discussion on the second order transport coefficients for R -charged black holes one should find the flow of Green's functions for two point correlation functions of R -currents. As we will mention in section VI that in presence of finite charges (or chemical potentials) it is very hard to solve the *Riccati* equation even perturbatively in ω and q . We find it very difficult to get any analytic solution for R -current Green's function. However we consider a simple model in section VI and study the flow of R -current Green's function numerically.

B. Charged black holes in higher derivative gravity

In this section, we will study five-dimensional gravity in presence of a negative cosmological constant and coupled to $U(1)$ gauge field. The model has been studied in [33, 50, 51], the action is given as,

$$\begin{aligned} S = & \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R + 12 - \frac{1}{4} F^2 \right. \\ & + \frac{\zeta}{3} \epsilon^{abcde} A_a F_{bc} F_{de} + \alpha' \left(c_1 R_{abcd} R^{abcd} \right. \\ & + c_2 R_{abcd} F^{ab} F^{cd} + c_3 (F^2)^2 + c_4 F^4 \\ & \left. \left. + c_5 \epsilon^{abcde} A_a R_{bcfg} R_{de}^{fg} \right) \right]. \end{aligned} \quad (\text{V.20})$$

Here, $F^2 = F_{ab} F^{ab}$, $F^4 = F_{ab} F^{bc} F_{cd} F^{da}$, and the AdS radius is set to unity. The action includes the Chern-Simon term and also a generic set of four derivative terms. All the four derivative terms will be treated perturbatively in our computation and here $\alpha' \ll 1$ is the perturbation parameter. In [33], it was shown that, within perturbative approach, after using field-redefinition, this is the most generic four derivative action that one can write down. In this section, we will closely follow their work. The background metric and the gauge field in presence of these higher derivative terms have the following form,

$$\begin{aligned} ds^2 &= -r^2 f(r) dt^2 + \frac{1}{r^2 g(r)} dr^2 + r^2 (dx^2 + dy^2 + dz^2), \\ A &= h(r) dt, \end{aligned} \quad (\text{V.21})$$

where,

$$\begin{aligned} f(r) &= f_0(r) (1 + \alpha' F(r)), \\ g(r) &= f_0(r) (1 + \alpha' (F(r) + G(r))), \\ h(r) &= h_0(r) + \alpha' H(r). \end{aligned} \quad (\text{V.22})$$

Here $f_0(r)$, $g_0(r)$ and $h_0(r)$ are the solution of the background in absence of the higher-derivative terms in the action and they are given as,

$$\begin{aligned} f_0 &= g_0 = \left(1 - \frac{r_0^2}{r^2} \right) \left(1 + \frac{r_0^2}{r^2} - \frac{Q^2}{r_0^2 r^4} \right), \\ h_0 &= \sqrt{3} Q \left(\frac{1}{r_0^2} - \frac{1}{r^2} \right). \end{aligned} \quad (\text{V.23})$$

Here, Q is related to the physical charge of the system and r_0 is the position of the horizon. From (V.22), it is clear that even in presence of the higher-derivative terms, the horizon remains at r_0 . The higher-derivative corrections to this background are given by the functions $F(r)$, $G(r)$ and $H(r)$. The form of these functions are given in [33]. We would not write those expressions and refer the reader to that paper.

Using the flow equation, we will study the higher order transport coefficient of the plasma theory dual to this gravity model. For this, we will write the effective action

for the metric fluctuation in (II.3), as we have done in previous sections,

$$S_{\text{eff}} = \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} dr \left[\mathcal{A}_1^{CB}(r, k) \phi'(r, k) \phi'(r, -k) + \mathcal{A}_0^{CB}(r, k) \phi(r, k) \phi(r, -k) \right] \quad (\text{V.24})$$

where, \mathcal{A}_1^{CB} and \mathcal{A}_0^{CB} are given in appendix E. The corresponding the flow equation (II.15) for this case with the coefficients \mathcal{A}_1^{CB} and \mathcal{A}_0^{CB} is,

$$\partial_r \bar{\chi}^{\text{CB}}(k_\mu, r) = i\omega \sqrt{-\frac{g_{rr}}{g_{tt}}} \left[\frac{\bar{\chi}^{\text{CB}}(k_\mu, r)^2}{\Sigma^{\text{CB}}(r, k)} - \frac{\Upsilon^{\text{CB}}(r, k)}{\omega^2} \right], \quad (\text{V.25})$$

where we define

$$\begin{aligned} \Sigma^{\text{CB}}(r, k) &= -2\mathcal{A}_1^{\text{CB}}(r, k_\mu) \sqrt{-\frac{g_{rr}}{g_{tt}}} \\ \Upsilon^{\text{CB}}(r, k) &= 2\mathcal{A}_0^{\text{CB}}(r, k_\mu) \sqrt{-\frac{g_{tt}}{g_{rr}}}. \end{aligned} \quad (\text{V.26})$$

We solve this flow equation to find the effect of higher derivative terms and chemical potential (or charge) on transport coefficient. Here, we present our result for small Q only, though it is possible to find the results for any Q .

The boundary condition (II.18) will take the following form¹⁷,

$$\bar{\chi}(k_\mu, r_0) = \frac{r_0^3}{16\pi G_5} \left[1 - \alpha' \frac{24c_1 Q^2}{r_0^6} \right]. \quad (\text{V.27})$$

In this case, the horizon value of the response function is independent of momenta. With this boundary condition, we can solve the flow equation (V.25), and the solution is given by,

$$\begin{aligned} i\omega \bar{\chi}(k_\mu, \infty) &= i\omega \frac{r_0^3}{16\pi G_5} \left[1 - \alpha' \frac{24c_1 Q^2}{r_0^6} \right] \\ &\quad - \omega^2 \frac{r_0^2}{32\pi G_5} \left[(1 - \ln 2) - \frac{Q^2}{2r_0^6} (3 - \ln 2) \right. \\ &\quad \left. + \frac{\alpha'}{6} \left(c_1 (2 - 5 \ln 2) - \frac{Q^2}{2r_0^6} (c_1 (35 - 58 \ln 2) - 48c_2 (2 - \ln 2)) \right) \right] \\ &\quad + \frac{q^2 r_0^2}{32\pi G_5} \left[1 - \frac{\alpha'}{3} \left(25c_1 + \frac{Q^2}{r_0^6} (32c_1 + 24c_2) \right) \right] \\ &\quad + \mathcal{O}(q\omega^2, \omega q^2, q^3, \omega^3) + \mathcal{O}(Q^4). \end{aligned} \quad (\text{V.28})$$

It is easy to read off the transport coefficients from this expressions.

$$\begin{aligned} \eta &= \frac{r_0^3}{16\pi G_5} \left[1 - \alpha' \frac{24c_1 Q^2}{r_0^6} \right] + \mathcal{O}(Q^4) \\ \kappa &= \frac{\eta}{\pi T} \left[\left(1 - \frac{Q^2}{2r_0^6} \right) - \alpha' \left(10c_1 - \frac{Q^2}{3r_0^6} (37c_1 - 48c_2) \right) \right] + \mathcal{O}(Q^4) \\ \tau_\pi T &= \frac{2 - \ln 2}{2\pi} - \frac{Q^2 (5 - 3 \ln 2)}{4\pi r_0^6} \\ &\quad + \alpha' \left[-\frac{11c_1}{2\pi} + \frac{Q^2}{4\pi r_0^6} (-16c_2 + 5c_1 (11 - 4 \ln 2)) \right] + \mathcal{O}(Q^4) \end{aligned} \quad (\text{V.29})$$

where, the temperature T of the system is given by,

$$T = \frac{r_0}{\pi} \left[\left(1 - \frac{Q^2}{2r_0^6} \right) - \frac{\alpha'}{3} \left(5c_1 + \frac{Q^2}{2r_0^6} (31c_1 + 48c_2) \right) \right] + \mathcal{O}(Q^4). \quad (\text{V.30})$$

We see that the first order as well as the second order transport coefficients coming from retarded Green's function of energy momentum tensor¹⁸ only depends on two coefficients c_1, c_2 . This feature was observed in [33] for entropy density s and first-order transport coefficients η . The coefficients c_3, c_4 , which parameterize couplings in the four point function of the dual $U(1)$ current does not play any role in these hydrodynamic coefficients. They should be important for the computation of conductivity, which comes from the Green's function of the boundary R -current. Two other coefficients ζ, c_5 also do not appear in the expressions. One can find a magnetic brane solution of the action (V.20) like [52]. In that case it would be interesting to find the effect of magnetic field on transport coefficients.

VI. FLOW OF RETARDED GREEN'S FUNCTION OF BOUNDARY R CURRENT

Finally, in this section, we study the flow of retarded Green's function of boundary R -current,

$$G_{i,j}^R(k) = -i \int dt d^3 x e^{ik \cdot x} \langle [J_i(x), J_j(0)] \rangle \quad (\text{VI.1})$$

where $J_\mu(x)$ is the CFT current dual to a bulk gauge field A_μ .

¹⁷ Note that we are working in a different coordinate where $r \rightarrow \infty$ is the boundary, therefore we choose the positive branch of the boundary condition (II.18)

¹⁸ It would be interesting to study the flow of retarded Green's function for boundary R -current. We found it to be difficult to get any analytical solution for response function in presence of finite chemical potential and higher derivative terms. However, it would be nice to know the higher derivative corrections to other second order transport coefficients appear in R current [47, 48].

In hydrodynamic approximation one can express the current in powers of boundary derivatives. Up to first order in derivative expansion it has the following form,

$$J_\nu = -\tilde{\kappa} P_\nu^\alpha \partial_\alpha \frac{\mu}{T} + \Omega l_\nu + \mathcal{O}(\partial^2) \quad (\text{VI.2})$$

where, $\tilde{\kappa}$ and Ω are two first order transport coefficients, μ is chemical potential, T is temperature and

$$\begin{aligned} P_{\mu\nu} &= u_\mu u_\nu + \eta_{\mu\nu} \\ l_\mu &= \epsilon_\mu^{\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma. \end{aligned} \quad (\text{VI.3})$$

The expression of J_μ up to second order in derivative expansion can be found in [47, 48]. From conformal invariance of the theory it is possible to write all possible second order transport coefficients appear in the expression of J_μ . However, like energy momentum tensor, from the expression of retarded Green's function it is not possible to compute all the transport coefficients that appear in different order of derivative expansion.

In this section we study the flow equation of retarded Green's function of boundary R -current. Unfortunately we find it difficult to solve the flow equation analytically to extract any transport coefficient. We present our calculation how to write the flow equation for retarded Green's function of R -currents in presence of generic higher derivative terms in bulk Lagrangian and some numerical results.

We start with Einstein-Maxwell action

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + 12 - \frac{1}{4} F^2 \right). \quad (\text{VI.4})$$

Solution is given by equation (V.21) with $\alpha' = 0$. The temperature of the black hole is given by,

$$T = \frac{r_0}{\pi} \left(1 - \frac{Q^2}{2r_0^6} \right) \quad (\text{VI.5})$$

and the chemical potential is given by,

$$\mu = \frac{\sqrt{3}Q}{r_0^2}. \quad (\text{VI.6})$$

For technical advantage we write the metric and gauge field in a different coordinate. We change the radial coordinate $r \rightarrow \frac{r_0}{\sqrt{r}}$. In this coordinate the metric and gauge field is given by,

$$\begin{aligned} ds^2 &= -\frac{r_0^2 U(r)}{r} dt^2 + \frac{dr^2}{4r^2 U(r)} + \frac{r_0^2}{r} (d\vec{x}^2) \\ A_t(r) &= E(r) \end{aligned} \quad (\text{VI.7})$$

where,

$$\begin{aligned} U(r) &= (1-r)(1+r - \frac{Q^2 r^2}{r_0^6}) \\ E(r) &= \frac{\sqrt{3}Q}{r_0^2} (1-r). \end{aligned} \quad (\text{VI.8})$$

We turn on small fluctuations for x component of gauge fields. Since the A_t component of the bulk vector is non-vanishing in this background, the perturbations A_x can couple to the tx component of graviton. Therefore we also need to consider small metric fluctuations for components g_{tx} . Writing them in momentum space

$$\begin{aligned} A_x(r, x) &= \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} A_1(r, k) \\ g_{tx}^x(r, x) &= \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} \Phi(r, k). \end{aligned} \quad (\text{VI.9})$$

However, there exists a constraint relation between A_x and g_{tx} . We use this relation to replace g_{tx} from equation of motion of A_x .

The on-shell action for gauge field fluctuations are given by,

$$\begin{aligned} S_A &= \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} \left[-r_0^2 U(r) A_x'^2(r, k) \right. \\ &\quad \left. + \left(\frac{\omega^2}{4rU(r)} - \frac{q^2}{4r} \right) A_x^2 \right]. \end{aligned} \quad (\text{VI.10})$$

The current corresponding to A_x fluctuation is given by,

$$\begin{aligned} J_x(r, k) &= \frac{\delta S_A}{\delta A_x'(r, k)} \\ &= -2r_0^2 U(r) A_x'(r, k). \end{aligned} \quad (\text{VI.11})$$

The equation of motion for $A_x(r, k)$ is given by,

$$\begin{aligned} (U(r) A_x'(r, k))' &= -\frac{1}{4r_0^2} \left(\frac{\omega^2}{rU(r)} - \frac{q^2}{r} \right) A_x(r, k) \\ &\quad - E'(r) \phi'(r, k) \\ &= -\frac{1}{4r_0^2} \left(\frac{\omega^2}{rU(r)} - \frac{q^2}{r} \right) A_x(r, k) \\ &\quad + \frac{rE'(r)^2}{r_0^2} A_x(r, k). \end{aligned} \quad (\text{VI.12})$$

Using the constraint relation (coming from rx component of Einstein equations),

$$\phi'(r, k) = -\frac{rE'(r)A_x(r, k)}{r_0^2}. \quad (\text{VI.13})$$

and the equations of motion one gets,

$$\begin{aligned} J_x'(r, k) &= \frac{1}{2} \left(\frac{\omega^2}{rU(r)} - \frac{q^2}{r} \right) A_x(r, k) \\ &\quad - 2rE'(r)^2 A_x(r, k). \end{aligned} \quad (\text{VI.14})$$

Next we define a response function

$$\sigma(r, k) = \frac{J_x(r, k)}{i\omega A_x(r, k)}. \quad (\text{VI.15})$$

Taking the derivative with respect to r and using the equation of motion we find the flow is given by,

$$\sigma'(r, k) = \frac{i\omega}{2r_0 U(r)} \left[\sigma(r, k)^2 - \left(\frac{r_0}{r} \left(1 - \frac{q^2 U(r)}{\omega^2} \right) - \frac{4r_0 r E'(r)^2 U(r)}{\omega^2} \right) \right]. \quad (\text{VI.16})$$

From the regularity of the response function at the horizon we find the boundary condition is given by,

$$\sigma(1, k)^2 = 1. \quad (\text{VI.17})$$

With this boundary condition one can integrate this nonlinear equation to find finite frequency response of boundary Green's function.

In presence of generic higher derivative terms in the Lagrangian the on-shell action for fluctuation A_x may not have canonical form like (VI.10). In that case one has to write an effective action for A_x like transverse graviton. The effective will have the same form as (VI.10) only the coefficients will depend on coupling constant of higher derivative terms.

We conclude this section by presenting some numerical solutions of the flow equation (VI.16) for both non-extremal extremal black holes in fig. 4.

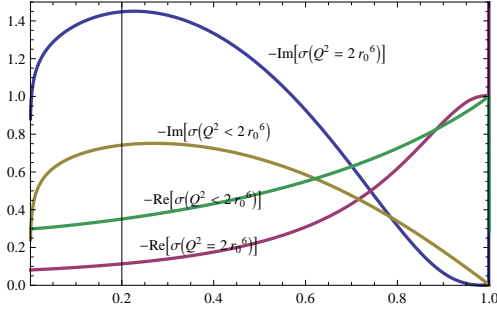


FIG. 4. Flow of σ for non-extremal and extremal black holes.

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Appendix A: Second order transport coefficients from Kubo formula

In this appendix we re-derive second order transport coefficients κ and τ_π from usual Kubo approach. Let us

now consider the action with solution given in section II,

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} [R + 12]. \quad (\text{A.1})$$

For well defined variation of this action we need to add a Gibbons-Hawking boundary term. Also, requiring the on-shell action being finite at boundary, we have to add counter-terms following usual approach of holographic renormalization. They are as follows:

$$S_{GH} = \frac{1}{8\pi G_5} \int d^4 x \sqrt{-\gamma} \mathcal{K},$$

$$S_{CT} = \frac{1}{16\pi G_5} \int d^4 x \sqrt{-\gamma} [6 + \frac{1}{2} \mathcal{R}] \quad (\text{A.2})$$

where γ and \mathcal{R} are boundary metric and Ricci scalar (constructed out of γ) respectively.

We consider the following metric perturbation,

$$g_{xy} = g_{xy}^{(0)} + h_{xy}(r, x) = g_{xy}^{(0)} (1 + \epsilon \Phi(r, x)), \quad (\text{A.3})$$

where ϵ is an order counting parameter. We are interested in quadratic on-shell action for this transverse graviton $\Phi(r, x)$. Let us define the Fourier transform,

$$\phi(r, k) = \int \frac{d^4 x}{(2\pi)^4} e^{-ik \cdot x} \Phi(r, x), \quad (\text{A.4})$$

and $k = \{-\omega, \vec{k}\}$. Substituting this fluctuation in action (A.1), we get (B.1)¹⁹. Now, we rewrite this action (A.1) as equation of motion piece (which vanishes on-shell) and boundary term²⁰. Thus, on-shell S, S_{GH}, S_{CT} become,

$$S = \frac{1}{16\pi G_5} \int_{r=\delta} \frac{d^4 k}{(2\pi)^4} L(\phi(r, k)),$$

$$S_{GH} = \frac{1}{8\pi G_5} \int_{r=\delta} \frac{d^4 k}{(2\pi)^4} L_{GH}(\phi(r, k)),$$

$$S_{CT} = \frac{1}{16\pi G_5} \int_{r=\delta} \frac{d^4 k}{(2\pi)^4} L_{CT}(\phi(r, k)), \quad (\text{A.5})$$

where we define,

$$L(\phi(r, k)) = a_2(r) \phi'(r, k) \phi^*(r, k) + \frac{a_4(r) - a_5'(r)}{2} \phi(r, k) \phi^*(r, k),$$

$$L_{GH}(\phi(r, k)) = 2(g_1) \phi(r, k) \phi^*(r, k) + 2(g_2) \phi'(r, k) \phi^*(r, k),$$

$$L_{CT}(\phi(r, k)) = (c_0 + c_1 \omega^2 + c_2 q^2) \phi(r, k) \phi^*(r, k). \quad (\text{A.6})$$

The coefficients a_2, a_4, a_5 are given in (B.2) and other coefficients are,

$$g_1 = \frac{2 - r^2}{r^2}, \quad g_2 = -2 \frac{1 - r^2}{r}$$

$$c_0 = -3 \frac{\sqrt{1 - r^2}}{r^2}, \quad c_1 = \frac{1}{4r \sqrt{1 - r^2}}$$

$$c_2 = -\frac{\sqrt{1 - r^2}}{4r}. \quad (\text{A.7})$$

¹⁹ a_3, a_6 are zero here

²⁰ We ignore contribution from horizon as [5]

The retarded Green's function is defined at asymptotic infinity as,

$$G^R(k) = 2 \lim_{r \rightarrow 0} \frac{(L + L_{GH} + L_{CT})|_{on-shell}}{\phi_0(k)\phi_0(-k)}. \quad (\text{A.8})$$

Here, $\phi_0(k)$ is the boundary value of the fluctuation (A.4). The retarded Green's function G^R is a function of boundary momenta $k_\mu = (\omega, 0, 0, q)$. Now the leading action and the Gibbons-Hawking action get divergences from $\phi'(r, k)\phi^*(r, k)$ and $\phi(r)\phi^*(r, k)$ parts. Both these divergences get canceled by the counter-term action which is always proportional to only $\phi(r)\phi^*(r, k)$. In this case of leading Einstein's gravity, it is even more simplified. Divergences coming from $\phi(r)\phi^*(r, k)$ piece of leading and Gibbons-Hawking action gets canceled by momentum independent piece of Counter-term action. It turns out that there is a cancellation among the corresponding coefficients as,

$$\lim_{r \rightarrow 0} \left(\frac{1}{2} (a_4(r) - a_5'(r)) + 2g_1 + c_0 \right) = \frac{1}{2}, \quad (\text{A.9})$$

i.e. the final contribution from $\phi(r)\phi^*(r, k)$ piece is only a finite number $\frac{1}{2}$. As the graviton fluctuation $\phi(r, \omega, \vec{k}) = \phi_0(1 + F(r, \omega, \vec{k}))$ and moreover $\lim_{r \rightarrow 0} F(r, \omega, \vec{k}) = 0$, we see that the $\phi(r)\phi^*(r)$ term above only contribute to pressure (the ω independent piece of G^R). It would never contribute to any transport coefficient.

Also, the divergences coming from $\phi'(r, k)\phi^*(r, k)$ piece of original and Gibbons-Hawking action, get canceled with the piece of the counter-term proportional to ω^2, q^2 (c_1 and c_2 are purely divergent at boundary). Here, the situation is more subtle, as there is no cancellation among the coefficients. One actually needs to put the solution of $\phi(r, \omega, \vec{k})$ to see the cancellation.

The overall lesson from this detailed analysis is that counter-term only cancel the UV divergences in usual holographic renormalization process and at most contribute to pressure of the boundary plasma. It has no effects on any transport coefficients. In [10], the author have computed second order transport coefficients for the plasmas dual to leading Einstein's gravity following this usual approach. The results are as follows,

$$\tau_\pi = \frac{2 - \ln 2}{2\pi T}, \quad \kappa = \frac{\eta}{\pi T}. \quad (\text{A.10})$$

These results match with the one we obtained in (II.28) by solving the flow equations.

Appendix B: Equivalence of Boundary Terms

In this appendix we will show explicitly why the transport coefficients computed from the original action and the effective action are same, even for any higher derivative theory. It was already noticed [29], that the two would give same first-order transport coefficient η with

a suitable choice of the overall normalization constant. Here, we show that, not just the first order transport coefficients, rather any higher order transport coefficients computed from the original action and the effective action are same.

We consider a general class of action for ϕ which appears when the higher derivative terms are made of different contraction of Ricci tensor, Riemann tensor, Weyl tensor, Ricci scalar etc. or their different powers. Since, all these tensors involve two derivatives of metric they can only have terms like $\partial_a \partial_b \Phi(r, x)$ and its lower derivatives. Therefor the most generic quadratic (in $\Phi(r, x)$, in linear response theory) action for this kind of higher derivative gravity has the following form (in momentum space)²¹

$$S = \frac{1}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} dr \left[a_1(r)\phi(r)^2 + a_2(r)\phi'(r)^2 + a_4(r)\phi(r)\phi'(r) + \alpha' a_6(r)\phi''(r)\phi'(r) + \alpha' a_3(r)\phi''(r)^2 + a_5(r)\phi(r)\phi''(r) \right] \quad (\text{B.1})$$

where,

$$\begin{aligned} a_1(r) &= \frac{-8r^2 + \omega^2 r + 8}{4r^3 - 4r^5} + \alpha' f_2(r) \\ a_2(r) &= -3r + \frac{3}{r} + \alpha' h_2(r) \\ a_4(r) &= -\frac{6}{r^2} - 2 + \alpha' g_2(r) \\ a_5(r) &= -4r + \frac{4}{r} + \alpha' j_2(r) \end{aligned} \quad (\text{B.2})$$

and $a_3(r), a_6(r), j_2(r), g_2(r), h_2(r)$ and $f_2(r)$ depends on higher derivative terms in the action and hence are computed purely from the background solution with $\alpha' \rightarrow 0$. Among these coefficients a_3 is special, as, it couples to ϕ''^2 . All four derivatives act on the graviton fluctuation and thus a_3 only depends on metric functions in (II.2) and there r-derivatives. It is easy to convince ourselves that $a_3 \propto r(r^2 - 1)^2 f(r, \alpha')$, where $f(r, \alpha')$ is a function that depend on the higher derivative terms and finite (constant or 0) at the boundary $r \rightarrow 0$. Now let us write the effective Lagrangian as follows,

$$S_{eff} = \frac{1 + \alpha' \Gamma}{16\pi G_5} \int \frac{d^4 k}{(2\pi)^4} dr \left[\frac{4r(r^2 - 1)^2 \phi'(r)^2 - \omega^2 \phi(r)^2}{4r^2(r^2 - 1)} + \alpha' \left(b_2(r)\phi(r)^2 + b_1(r)\phi'(r)^2 \right) \right]. \quad (\text{B.3})$$

Demanding that the equation of motion (up to order α') of ϕ derived from the original action and the above action

²¹ In all the expressions we have omitted k dependence of ϕ .

are same we get,

$$b1(r) = \frac{1}{2r(r^2-1)^2} [(-4r^3 - 12r + \omega^2)a3(r) + (r^2-1)(2\kappa r^4 - a6'(r)r^3 - 4\kappa r^2 + 2a3'(r)r^2 + 2(r^2-1)h2(r)r - 2(r^2-1)j2(r)r + a6'(r)r + 2\kappa + 2a3'(r))] \quad (B.4)$$

$$b2(r) = -\frac{1}{16r^2(r^2-1)^4} \left[(\omega^4 + 144r^3\omega^2)a3(r) + 4(r^2-1) \left(-4r^2f2(r)(r^2-1)^3 + ((\omega^2\kappa - 2r^2(r^2-1)j2''(r))(r^2-1) + 2r^2g2'(r)(r^2-1)^2 + r\omega^2a3''(r))(r^2-1) + (1-11r^2)\omega^2a3'(r) \right) \right] \quad (B.5)$$

The boundary terms coming from the original action (after adding *Gibbons-Hawking* boundary terms) are given by ²²,

$$S^B = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[-\frac{\phi(r)^2}{r^2} + \phi(r)^2 + r\phi'(r)\phi(r) - \frac{\phi'(r)\phi(r)}{r} + \alpha' \left(\frac{1}{2}g2(r)\phi(r)^2 - \frac{1}{2}j2'(r)\phi(r)^2 + (h2(r) - j2(r) - \frac{a6'(r)}{2})\phi'(r)\phi(r) + \frac{a3'(r)(\phi(r)\omega^2 + 4(r^4-1)\phi'(r))\phi(r)}{4r(r^2-1)^2} - \frac{a3(r)(6r\phi(r)\phi'(r)\omega^2)}{4r(r^2-1)^3} - \frac{a3(r)((r^2-1)(8r^3+24r-\omega^2)\phi'(r))\phi(r)}{4r(r^2-1)^3} - \frac{a3(r)\phi'(r)(\phi(r)\omega^2 + 4(r^4-1)\phi'(r))}{4r(r^2-1)^2} - a3(r)\phi'(r) \left(-\frac{\phi(r)\omega^2}{2r(r^2-1)^2} - \frac{(r^4-1)\phi'(r)}{r(r^2-1)^2} \right) \right] \quad (B.6)$$

And the boundary terms coming from the effective action

are given by,

$$S_{eff}^B = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[\left(r - \frac{1}{r} \right) \phi(r)\phi'(r) + \frac{\alpha'}{2r(r^2-1)^2} \left(\phi(r)(2\Gamma(r^2-1)^3 + (-a6'(r)r^3 + 2a3'(r)r^2 + 2(r^2-1)h2(r)r - 2(r^2-1)j2(r)r + a6'(r)r + 2a3'(r))(r^2-1) + (-4r^3 - 12r + \omega^2)a3(r))\phi'(r) \right) \right] \quad (B.7)$$

Now, it is interesting to compute the difference between these two boundary terms and the result is²³,

$$S^B - S_{eff}^B = \frac{1}{16\pi G_5} \int \frac{d^4k}{(2\pi)^4} \left[-\frac{\phi(r)^2}{r^2} + \phi(r)^2 + \alpha' \left(\frac{1}{2}g2(r)\phi(r)^2 - \frac{1}{2}j2'(r)\phi(r)^2 + \frac{a3'(r)\omega^2\phi(r)^2}{4r(r^2-1)} - \frac{a3(r)(6r\omega^2)\phi(r)^2}{4r(r^2-1)^3} \right) \right] \quad (B.8)$$

The term proportional to $a3$ in the parenthesis of (B.8) vanishes at the boundary whereas the term proportional to $a3'$ gives a pure UV divergent piece and a vanishing piece, due to the property of $a3$ mentioned above. This is true irrespective of the choice of the higher derivative terms. Thus, we see that the two boundary terms differ only by terms which are either purely divergent or of the form $g(r)\phi^2$, where $g(r)$ is any function of r . The divergent terms would get canceled once appropriate boundary terms are added (which has been discussed in sections II and III). The terms proportional to ϕ^2 can only contribute to pressure of the boundary theory and are not important for the computation of transport coefficients of the boundary plasma. Thus we see that, it is obvious that the transport coefficients coming from the original action and the boundary action are same.

Here we have considered only $R^{(n)}$ gravity theory. A more rigorous proof is required for theories involving covariant derivatives of curvature tensors and scalars.

²² There was a sign error in [29]

²³ It has been shown in [29] that $\Gamma = 0$.

Appendix C: Functions appeared in String theory corrected action

$$\begin{aligned} \mathcal{A}_0^{W^4}(r, k) = & -\frac{r_0^2(q^2(r^2-1) + \omega^2)}{4r^2(r^2-1)} \\ & -\frac{1}{4(r^2-1)} \left(\gamma(4q^2r^2((245r^4 \right. \\ & \left. -407r^2 + 162)r_0^2 + 10r\omega^2) \right. \\ & \left. +3(221r^4 - 191r^2 + 25)\omega^2r_0^2) \right) \end{aligned} \quad (\text{C.1})$$

and

$$\begin{aligned} \mathcal{A}_1^{W^4}(r, k) = & \frac{(r^2-1)r_0^4}{r} \\ & +3r(r^2-1)\gamma r_0^2 \left((43r^4 \right. \\ & \left. +47r^2 - 25)r_0^2 + 16q^2r^3 \right). \end{aligned} \quad (\text{C.2})$$

Appendix D: Functions appeared in four derivative action

$$\begin{aligned} \mathcal{A}_0^{\text{GB}}(r, k) = & -\frac{q^2(r^2-1) + \omega^2}{4r^2(r^2-1)} \\ & +\frac{\alpha'}{12r^2(r^2-1)} \left(q^2(2\beta_3(13r^2 - 3r\omega^2 - 13) \right. \\ & +130(r^2-1)\beta_1 + (-36r^4 + 25r^2 + 11)\beta_2) \\ & \left. +\omega^2((6r^2-11)\beta_2 + 130\beta_1 + 26\beta_3) \right) \end{aligned} \quad (\text{D.1})$$

and

$$\begin{aligned} \mathcal{A}_1^{\text{GB}}(r, k) = & r - \frac{1}{r} \\ & -\frac{(r^2-1)((18r^2-13)\beta_2 + 110\beta_1 + 22\beta_3)\alpha'}{3r}. \end{aligned} \quad (\text{D.2})$$

Appendix E: Functions appeared in Higher-derivative Charged black-hole action

$$\begin{aligned} \mathcal{A}_1^{CB} = & \frac{r^5(\omega^2 - q^2) + q^2r_0^4r}{2(r^4 - r_0^4)} \\ & +\frac{c_1\alpha'(11r^8(\omega^2 - q^2) - r_0^4r^4(25q^2 + 6\omega^2) + 36q^2r_0^8)}{6r^3(r^4 - r_0^4)} \\ & +\frac{Q^2}{2r_0^2(r^2 - r_0^2)(r^2 + r_0^2)^2} \left[r^3\omega^2 + \frac{\alpha'}{3r^5} \left(c_1[28q^2r_0^6r^2 \right. \right. \\ & \left. \left. - r^8(36q^2 + 127\omega^2) - 4r_0^2r^6(7q^2 - 3\omega^2) - 8q^2r_0^8 \right. \right. \\ & \left. \left. + 4r_0^4r^4(11q^2 + 3\omega^2) \right] - 24c_2[r_0^2r^6(2q^2 - 3\omega^2) - 2q^2r_0^6r^2 \right. \\ & \left. \left. + r_0^4r^4(2q^2 - 3\omega^2) - 2q^2r_0^8 + 4r^8\omega^2 \right] \right] + \mathcal{O}(Q^4) \end{aligned} \quad (\text{E.1})$$

and

$$\begin{aligned} \mathcal{A}_1^{CB} = & \frac{1}{2}(rr_0^4 - r^5) - \frac{c_1\alpha'(13r^4 - 18r_0^4)(r^4 - r_0^4)}{6r^3} \\ & +\frac{Q^2(r^2 - r_0^2)}{2rr_0^2} \left[1 - \frac{\alpha'}{3r^4} \left(24c_2(4r^4 - 3r_0^2r^2 - 3r_0^4) \right. \right. \\ & \left. \left. +c_1(101r^4 - 156r_0^2r^2 - 120r_0^4) \right) \right] + \mathcal{O}(Q^4). \end{aligned} \quad (\text{E.2})$$

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